

# On convexified packing and entropy duality

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**1. Introduction.** If  $K$  and  $B$  are subsets of a vector space (or just a group, or even a homogeneous space), the *covering number* of  $K$  by  $B$ , denoted  $N(K, B)$ , is the minimal number of translates of  $B$  needed to cover  $K$ . Similarly, the *packing number*  $M(K, B)$  is the maximal number of disjoint translates of  $B$  by elements of  $K$ . The two concepts are closely related; we have  $N(K, B - B) \leq M(K, B) \leq N(K, (B - B)/2)$ . If  $B$  is a ball in a normed space (or in an appropriate invariant metric) and  $K$  a subset of that space (the setting and the point of view we will usually employ), these notions reduce to considerations involving the smallest  $\epsilon$ -nets or the largest  $\epsilon$ -separated subsets of  $K$ .

Besides the immediate geometric framework, packing and covering numbers appear naturally in numerous subfields of mathematics, ranging from classical and functional analysis through probability theory and operator theory to information theory and computer science (where a code is typically a packing, while covering numbers quantify the complexity of a set). As with other notions touching on convexity, an important role is played by considerations involving duality. The central problem in this area is the 1972 “duality conjecture for covering numbers” due to Pietsch which has been originally formulated in the operator-theoretic context, but which in the present notation can be stated as

**Conjecture 1** *Do there exist numerical constants  $a, b \geq 1$  such that for any dimension  $n$  and for any two symmetric convex bodies  $K, B$  in  $\mathbb{R}^n$  one has*

$$b^{-1} \log N(B^\circ, aK^\circ) \leq \log N(K, B) \leq b \log N(B^\circ, a^{-1}K^\circ)? \quad (1)$$

Above and in what follows  $A^\circ := \{u \in \mathbb{R}^n : \sup_{x \in A} \langle x, u \rangle \leq 1\}$  is the polar body of  $A$ ; “symmetric” is a shorthand for “symmetric with respect to the origin” and, for definiteness, all logarithms are to the base 2. In our preferred setting of a normed space  $X$ , the proper generality is achieved by considering  $\log N(K, tB)$  for  $t > 0$ , where  $B$  is the unit ball and  $K$  – a generic (convex, symmetric) subset of  $X$ . The polars should then be thought of as subsets of  $X^*$ , with  $B^\circ$  the unit ball of that space. With minimal care, infinite-dimensional spaces and sets may be likewise considered. To avoid stating boundedness/compactness hypotheses, which are peripheral to the phenomena in question, it is convenient to allow  $N(\cdot, \cdot)$ ,  $M(\cdot, \cdot)$  etc. to take the value  $+\infty$ .

The quantity  $\log N(K, tB)$  has a clear information-theoretic interpretation: it is the complexity of  $K$ , measured in bits, at the level of resolution  $t$  with respect to the metric for which  $B$  is the unit ball. Accordingly, (1) asks whether the complexity of  $K$  is controlled by that of the ball in the dual space with respect to  $\|\cdot\|_{K^\circ}$  (the gauge of  $K^\circ$ , i.e., the norm whose unit ball is  $K^\circ$ ), and *vice versa*, at *every* level of resolution. [The original formulation of the conjecture involved

relating – in a quantitative way – compactness of an operator to that of its adjoint.] In a very recent paper [1] Conjecture 1 has been verified in the special yet most important case where  $B$  is an ellipsoid (or, equivalently, when  $K$  is a subset of a Hilbert space); the reader is referred to that article for a more exhaustive discussion of historical and mathematical background and for further references.

In the present note we introduce a new notion, which we call “convex separation” (or “convexified packing”) and prove a duality theorem related to that concept. This will lead to a generalization of the results from [1] to the setting requiring only mild geometric assumptions about the underlying norm. [Both the definition and the generalization are motivated by an earlier paper [2].] For example, we now know that Conjecture 1 holds – in the sense indicated in the paragraph following (1) – in all  $\ell_p$ - and  $L_p$ -spaces (classical or non-commutative) for  $1 < p < \infty$ , with constants  $a, b$  depending only on  $p$  and uniformly bounded if  $p$  stays away from 1 and  $\infty$ , and similarly in all uniformly convex and all uniformly smooth spaces.

At the same time, and perhaps more importantly, the new approach “demystifies” duality results for usual covering/packing numbers in that it splits the proof into two parts. One step is a duality theorem for convex separation, which predictably is a consequence of the Hahn-Banach theorem. The other step involves geometric considerations relating convex separation to the usual separation; while often delicate and involved, they are always set in a given normed space and reflect properties of that space without appealing to duality, and thus are conceptually simpler.

In the next two sections we shall give the definition of convex separation and prove the corresponding duality theorem. Then, in section 4, we shall state several estimates for the convexified packing/convex separation numbers, in particular those that compare them to the usual packing/covering numbers. In section 5 we state the generalization of the duality result from [1] alluded to above, and give some hints at its proof. We include details only for the proof of Theorem 2 (duality for convex separation) which is the part we consider conceptually new. Although we see our Theorem 5 as an essential progress towards Conjecture 1, in this announcement we omit the proofs; while non-trivial and technically involved, they require tools which were developed and used in our previous papers.

**2. Defining convex separation.** The following notion plays a central role in this note. For a set  $K$  and a symmetric convex body  $B$  we define

$$\hat{M}(K, B) := \sup\{N : \exists x_1, \dots, x_N \in K \text{ such that} \\ (x_j + \text{int}B) \cap \text{conv}\{x_i, i < j\} = \emptyset\}, \quad (2)$$

where “int” stands for the interior of a set.<sup>1</sup> We shall refer to any sequence satisfying the condition (2) as *B-convexly separated*. Leaving out the convex hull operation “conv” leads to the usual  $B$ -separated set, which is the same as  $B/2$ -packing. Thus we have  $\hat{M}(K, B) \leq M(K, B/2)$ . We emphasize that, as opposed to the usual notions of packing and covering, the *order* of the points is important here.

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<sup>1</sup>When defining packing in convex geometry, it is customary to require that only the *interiors* of the translates of  $B$  be disjoint; we follow that convention here even though it is slightly unsound in the categorical sense.

The definition (2) is very natural from the point of view of complexity theory and optimization. A standard device in constructing geometric algorithms is a “separation oracle” (cf. [3]): if  $T$  is a convex set then, for a given  $x$ , the oracle either attests that  $x \in T$  or returns a functional efficiently separating  $x$  from  $T$ . It is arguable that quantities of the type  $\hat{M}(T, \cdot)$  correctly describe complexities of the set  $T$  with respect to many such algorithms.

Since, as pointed out above and in the preceding section, packings, coverings and separated sets are very closely connected, and the corresponding “numbers” are related via two sided estimates involving (at most) small numerical constants, in what follows we shall use all these terms interchangeably.

**3. Duality for  $\hat{M}(K, B)$ .** While it is still an open problem whether Conjecture 1 holds in full generality, the corresponding duality statement for convex separation is fairly straightforward.

**Theorem 2** *For any pair of convex symmetric bodies  $K, B \subset \mathbb{R}^n$  one has*

$$\hat{M}(K, B) \leq \hat{M}(B^\circ, K^\circ/2)^2.$$

*Proof of Theorem 2* Let  $R := \sup\{\|x\|_B : x \in K\}$ ; i.e.,  $R$  is the radius of  $K$  with respect to the gauge of  $B$ . We will show that

- (i)  $\hat{M}(B^\circ, K^\circ/2) \geq \hat{M}(K, B)/\lceil 4R \rceil$
- (ii)  $\hat{M}(B^\circ, K^\circ/2) \geq \lfloor 4R \rfloor + 1$

Once the above are proved, Theorem 2 readily follows. To show (i), denote  $N = \hat{M}(K, B)$  and let  $x_1, \dots, x_N$  be a  $B$ -convexly separated sequence in  $K$ . Then, by (the elementary version of) the Hahn-Banach theorem, there exist separating functionals  $y_1, \dots, y_N \in B^\circ$  such that

$$1 \leq i < j \leq N \Rightarrow \langle y_j, x_j - x_i \rangle = \langle y_j, x_j \rangle - \langle y_j, x_i \rangle \geq 1, \quad (3)$$

a condition which is in fact equivalent to  $(x_j)$  being  $B$ -convexly separated. Now  $x_j \in K \subset RB$  and  $y_j \in B^\circ$  imply that  $-R \leq \langle y_j, x_j \rangle \leq R$ , and hence dividing  $[-R, R]$  into  $\lceil 4R \rceil$  subintervals of length  $\leq 1/2$  we may deduce that one of these subintervals contains  $M \geq N/\lceil 4R \rceil$  of the numbers  $\langle y_j, x_j \rangle$ . To simplify the notation, assume that this occurs for  $j = 1, \dots, M$ , that is

$$1 \leq i, j \leq M \Rightarrow -1/2 \leq \langle y_i, x_i \rangle - \langle y_j, x_j \rangle \leq 1/2. \quad (4)$$

Combining (3) and (4) we obtain for  $1 \leq i < j \leq M$

$$\langle y_i - y_j, x_i \rangle = \langle y_i, x_i \rangle - \langle y_j, x_j \rangle + \langle y_j, x_j \rangle - \langle y_j, x_i \rangle \geq -1/2 + 1 = 1/2,$$

which is again a condition of type (3) and thus shows that the sequence  $y_M, \dots, y_1$  (in this order!) is  $K^\circ/2$ -convexly separated. Hence  $\hat{M}(B^\circ, K^\circ/2) \geq M \geq N/\lceil 4R \rceil$ , which is exactly the conclusion of (i).

To show (ii) we note that  $R$  is also the radius of  $B^\circ$  with respect to the gauge of  $K^\circ$ . Since we are in a finite-dimensional space, that radius is attained and so there is a segment  $I := [-y, y] \subset B^\circ$  with  $\|y\|_{K^\circ} = R$ . This implies that  $M(I, K^\circ/2) \geq \lfloor 4R \rfloor + 1$ . However, in dimension one separated and convexly separated sets coincide; this allows to conclude that  $\hat{M}(B^\circ, K^\circ/2) \geq \hat{M}(I, K^\circ/2) \geq \lfloor 4R \rfloor + 1$ , as required.

**4. Separation vs. convex separation.** It appears at the first sight that the notion of convex separation is much more restrictive than that of usual separation and, consequently, that – except for very special cases such as that of one-dimensional sets mentioned above –  $\hat{M}(\cdot, \cdot)$  should be significantly smaller than  $M(\cdot, \cdot)$  or  $N(\cdot, \cdot)$ . However, we do not have examples when that happens. On the other hand, for several interesting classes of sets we do have equivalence for not-so-trivial reasons. Here we state two such results.

**Theorem 3** *There exist numerical constants  $C, c > 0$  such that, for any  $n \in \mathbb{N}$  and any pair of ellipsoids  $\mathcal{E}, \mathcal{B} \subset \mathbb{R}^n$  one has*

$$\log M(\mathcal{E}, \mathcal{B}) \leq C \log \hat{M}(\mathcal{E}, c\mathcal{B}).$$

While Theorem 3 deals with purely Euclidean setting, the next result holds under rather mild geometric assumptions about the underlying norm. It requires  $K$ -convexity, a property which goes back to [5] and is well known to specialists; we refer to [8] for background and properties. While many interesting descriptions of that class are possible, here we just mention that  $K$ -convexity is equivalent to the absence of large subspaces well-isomorphic to finite-dimensional  $\ell_1$ -spaces and that it can be quantified, i.e., there is a parameter called the  $K$ -convexity constant and denoted  $K(X)$ , which can be defined both for finite and infinite dimensional spaces, and which has good permanence properties with respect to standard functors of functional analysis. For example, as hinted in the introduction, all  $L_p$ -spaces for  $1 < p < \infty$  are  $K$ -convex (with constants depending only on  $p$ ), and similarly all uniformly convex and all uniformly smooth spaces.

**Theorem 4** *Let  $X$  be a normed space which is  $K$ -convex with  $K(X) \leq \kappa$  and let  $B$  be its unit ball. Then for any bounded set  $T \subset X$  one has*

$$\log M(T, B) \leq \beta \log \hat{M}(T, B/2),$$

where  $\beta$  depends only on  $\kappa$  and the diameter of  $T$ . Similarly, if  $r > 0$  and if  $U$  is a symmetric convex subset of  $X$  with  $U \supset rB$ , then

$$\log M(B, U) \leq \beta' \log \hat{M}(B, U/2)$$

with  $\beta'$  depending only on  $\kappa$  and  $r$ .

The proof of Theorem 3 is non-trivial but elementary. The proof of Theorem 4 is based on the so called Maurey's lemma (see [7]) and the ideas from [2]. The details of both arguments will be presented elsewhere.

**5. Duality of covering and packing numbers in  $K$ -convex spaces.** If  $B$  is the unit ball in a  $K$ -convex space  $X$ , then, combining Theorems 2 and 4, we obtain for any bounded symmetric convex set  $T \subset X$

$$\log M(T, B) \leq \beta \log \hat{M}(T, B/2) \leq 2\beta \log \hat{M}(B^\circ, T^\circ/4) \leq 2\beta \log M(B^\circ, T^\circ/8),$$

where  $\beta$  depends only on the diameter of  $T$  (and on  $K(X)$ ) and, similarly,  $\log M(B^\circ, T^\circ) \leq 2\beta' \log M(T, B/8)$ . To show the latter, we apply the second part of Theorem 4 to

$X^*$ , the dual of  $X$ , and to  $U = T^\circ \supset (2/\text{diam} T)B^\circ$  (see the comments following (1)), and use the known fact that  $K(X^*) = K(X)$ .

On the other hand, an iteration scheme developed in [1] can be employed to show that if, for some normed space  $X$ , duality in the sense of the preceding paragraph holds – with some constants  $\beta, \beta'$  – for, say, all  $T \subset 4B$ , then it also holds for *all* sets  $T \subset X$  with constants depending only on  $\beta, \beta'$ . We thus have

**Theorem 5** *Let  $X$  be a normed space which is  $K$ -convex with  $K(X) \leq \kappa$  and let  $B$  be its unit ball. Then for any symmetric convex set  $T \subset X$  and any  $\epsilon > 0$  one has*

$$b^{-1} \log M(B^\circ, a\epsilon T^\circ) \leq \log M(T, \epsilon B) \leq b \log M(B^\circ, a^{-1}\epsilon T^\circ),$$

where  $a, b \geq 1$  depend only on  $\kappa$ .

Theorem 5 is related to [1] in a similar way as the results of [2] were related to [10]: in both cases a statement concerning duality of covering numbers is generalized from the Hilbertian setting to that of a  $K$ -convex space. A more detailed exposition of its proof will be presented elsewhere.

**6. Final remarks.** While the approach via convexly separated sets does not yet settle Conjecture 1 in full generality, it includes, in particular, all cases for which the Conjecture has been previously verified. One such special case, which does not follow directly from the results included in the preceding sections, was settled in [4] and subsequently generalized in [9]: *If, for some  $\gamma > 0$ ,  $\log N(K, B) \geq \gamma n$ , then  $\log N(B^\circ, K^\circ/a) \geq b^{-1} \log N(K, B)$  with  $a, b \geq 1$  depending only on  $\gamma$ . In the same direction, we have*

**Theorem 6** *Let  $\gamma > 0$  and let  $K, B \subset \mathbb{R}^n$  be symmetric convex bodies with  $\log M(K, B) \geq \gamma n$ . Then*

$$\log \hat{M}(K, B/\alpha) \geq \beta^{-1} \log M(K, B), \quad (5)$$

where  $\alpha \geq 1$  depends only on  $\gamma$  and  $\beta \geq 1$  is a universal constant.

The result from [4], [9] mentioned above is then an easy corollary: combine (5) with Theorem 2 to obtain  $\log M(K, B) \leq 2\beta \log \hat{M}(B^\circ, (2\alpha)^{-1}K^\circ)$ , and the later expression is “trivially”  $\leq 2\beta \log M(B^\circ, (4\alpha)^{-1}K^\circ)$ . In turn, Theorem 6 can be derived by applying Theorem 3 to the so-called  $M$ -ellipsoids of  $K$  and  $B$ . This is admittedly not the simplest argument, but it subscribes to our philosophy of minimizing the role of duality considerations and sheds additional light on the relationship between  $M(\cdot, \cdot)$  and  $\hat{M}(\cdot, \cdot)$ .

We conclude the section and the note by pointing out that a slightly different – but equally natural from the algorithmic point of view – definition of convex separation is possible. Namely, we may consider sets of points in  $K$  verifying (2) with a modified condition  $(x_j + \text{int} B) \cap \text{conv}\{x_i, i \neq j\} = \emptyset$ . For the so modified convex separation number – let us call it  $\tilde{M}(K, B)$  – the statement (i) from the proof of Theorem 2 remains true (by an almost identical argument), and so for “bounded” sets we do have duality. However, the statement (ii) is false already for the simple case of a segment.

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